

FINAL STATE PROBLEM FOR THE CUBIC NONLINEAR SCHRÖDINGER EQUATION WITH REPULSIVE DELTA POTENTIAL

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Abstract. We consider the asymptotic behavior in time of solutions to the cubic nonlinear Schrödinger equation with repulsive delta potential (δ -NLS). We shall prove that for a given asymptotic profile u_{ap} , there exists a solution u to (δ -NLS) which converges to u_{ap} in $L^2(\mathbb{R})$ as $t \rightarrow \infty$. To show this result we exploit the distorted Fourier transform associated to the Schrödinger equation with delta potential.

1. INTRODUCTION

We consider the nonlinear Schrödinger equation with delta potential:

$$i\partial_t u + \frac{1}{2}\partial_x^2 u - q\delta u = \lambda|u|^2 u, \quad t, x \in \mathbb{R}, \quad (1.1)$$

where $q \neq 0$ and $\lambda \neq 0$ are real constants, and δ is the Dirac measure at the origin.

The cubic nonlinear Schrödinger equation with localized potential provides a simpler model describing the resonant nonlinear propagation of light through optical wave guides with localized defects, see Goodman-Holmes-Weinstein [9]. The delta potential is called “repulsive” for $q > 0$ and “attractive” for $q < 0$.

Along with the nonlinear Schrödinger equation, (1.1) has a standing wave solution. The stability of the soliton is studied in Goodman-Holmes-Weinstein [9], Fukuizumi-Ohta-Ozawa [7] by the variational arguments, and Deift-Park [6] via the nonlinear steepest-descent method for Riemann-Hilbert problems.

Eq. (1.1) is currently being intensively investigated in the point of view of the interaction between the soliton and the delta potential [5, 9, 12, 13, 14]. Holmer-Zworski [14] studied the behavior of slow solitons under the flow by (1.1). Later on, Holmer-Marzuola-Zworski [12, 13] and Datchev-Holmer [5] have studied the splitting of the fast solitons by the delta potential. It was shown that the high velocity incoming soliton is split into a transmitted component and a reflected component for both repulsive [13] and attractive [5] cases, see [9, 12] for the numerical results.

In this paper we study the long time behavior of solutions to (1.1). More precisely, we shall prove that for a given asymptotic profile u_{ap} , there exists a solution u to (1.1) which

1991 *Mathematics Subject Classification.* Primary 35Q51, 35P25; Secondary 37K05, 37K40.

Key words and phrases. Schrödinger equation with delta potential, asymptotic behavior.

converges to u_{ap} in $L^2(\mathbb{R})$ as $t \rightarrow \infty$. So we consider the final state problem (1.1) and

$$\lim_{t \rightarrow +\infty} (u(t) - u_{ap}) = 0 \quad \text{in } L^2. \quad (1.2)$$

Although we consider the behavior of solution to (1.1) as $t \rightarrow \infty$, we can treat behavior of solution as $t \rightarrow -\infty$ in a similar fashion.

Before we state our result, we summarize known results for the scattering and asymptotic behavior of solution to the nonlinear Schrödinger equation. There is a large literatures on the scattering theory of the solution to nonlinear Schrödinger equation starting with the pioneering work of Lin-Strauss [16] and Ginibre-Velo [8]. We are concerned here only with the one dimensional case.

Let us consider the nonlinear Schrödinger equation with a potential

$$i\partial_t v + \frac{1}{2}\partial_x^2 v - V(x)v = \mu|v|^{p-1}v, \quad t, x \in \mathbb{R}, \quad (1.3)$$

where $p > 1$, $\mu \neq 0$ and V is a real valued function. For the case $V \equiv 0$, it is known that the power $p = 3$ is the borderline between the short and long range scattering theory, namely, the solution to (1.3) scatters to the solution to the free Schrödinger equation for $p > 3$ and no solution of (1.3) has a scattering state for $p \leq 3$, see Barab [2] and Tsutsumi-Yajima [19]. Therefore the solution of the nonlinear equation (1.1) with $p \leq 3$ may be have a different asymptotic profile from the solution of the free Schrödinger equation.

In this regard, Ozawa [17] proved that for a given small final data ϕ_+ , there exists a solution v to (1.3) with $p = 3$ satisfying

$$v(t) \sim v_{ap}(t) = t^{-1/2} \mathcal{F}[\phi_+]\left(\frac{x}{t}\right) \exp\left(\frac{ix^2}{2t} - i\lambda|\mathcal{F}[\phi_+]\left(\frac{x}{t}\right)|^2 \log t - i\frac{\pi}{4}\right) \quad (1.4)$$

as $t \rightarrow +\infty$, where \mathcal{F} is the Fourier transform. Note that v_{ap} is the leading term of $e^{(i/2)t\partial_x^2}\phi_+$ with the phase modification. Later on, Hayashi-Naumkin [10] proved that for a given small initial data v_0 , there exists a unique solution v to (1.3) with $p = 3$ satisfying $v(0, x) = v_0(x)$ and (1.4) for some ϕ_+ . See also Hayashi-Naumkin [11] for the improvement of those results.

For the case where $V \not\equiv 0$, Cuccagna-Georgiev-Visciglia [4] have recently shown that the solution to (1.3) scatters to the solution of the free Schrödinger equation under the suitable assumptions on V and $p > 3$. As far as we know, there is no result on the long range scattering for (1.3) with $V \not\equiv 0$.

We briefly explain why the critical exponent p is three for the one dimensional Schrödinger equation. Roughly speaking, the nonlinearity of (1.3) is short range if L^2 norm of the nonlinear term is integrable on time interval $[1, \infty)$. Since the pointwise decay of the solution to the one dimensional free Schrödinger equation is $\mathcal{O}(t^{-1/2})$ as $t \rightarrow \infty$, the L^2 norm of the nonlinear term $|v|^{p-1}v$ decays like $\mathcal{O}(t^{-(p-1)/2})$. Since the integral $\int_1^\infty t^{-(p-1)/2} dt$ is finite if and only if $p > 3$, the exponent $p = 3$ appears as the borderline between the short and long range scattering theory. For further results on the scattering theory on the nonlinear Schrödinger equation, see e.g., Cazenave [3, Chapter 7].

Let us return to the equation (1.1). As with (1.3), the pointwise decay of the solution to the free Schrödinger equation with delta potential is $\mathcal{O}(t^{-1/2})$, see proof of Lemma 2.2

below. Hence we expect that (1.1) fall into the long range case. In fact, we obtain the following results for the final state problem (1.1)-(1.2).

Theorem 1.1. *Let $q \geq 0$. There exists $\varepsilon > 0$ and $C > 0$ with the following properties: For any $\phi_+ \in \mathcal{S}'(\mathbb{R})$ with $\|(1 + |x|)\phi_+\|_{L_x^2} < \varepsilon$ there exists a unique global solution $u \in C(\mathbb{R}; L_x^2(\mathbb{R})) \cap L_{loc}^4(\mathbb{R}; L_x^\infty(\mathbb{R}))$ to (1.1) satisfying*

$$\sup_{t \geq 1} t^\alpha \left\{ \|u(t) - u_{ap}(t)\|_{L_x^2} + \left(\int_t^\infty \|u(\tau) - u_{ap}(\tau)\|_{L_x^\infty}^4 d\tau \right)^{1/4} \right\} \leq C,$$

where $1/4 < \alpha < 1/2$ and u_{ap} is given by

$$u_{ap}(t, x) = t^{-1/2} \mathcal{F}_q[\phi_+](\frac{x}{t}) \exp \left(\frac{ix^2}{2t} - i\lambda |\mathcal{F}_q[\phi_+](\frac{x}{t})|^2 \log t - i\frac{\pi}{4} \right)$$

and \mathcal{F}_q is a distorted Fourier transform associated to Schrödinger operator with delta potential:

$$\mathcal{F}_q[\phi](\xi) = \begin{cases} \mathcal{F}[\phi(x) - q1_-(x)e^{qx} \int_x^{-x} e^{q|y|}\phi(y)dy](\xi) & \text{if } \xi \geq 0, \\ \mathcal{F}[\phi(x) - q1_+(x)e^{-qx} \int_{-x}^x e^{q|y|}\phi(y)dy](\xi) & \text{if } \xi < 0. \end{cases}$$

Remark 1.1. In this paper we consider the repulsive case only. The reason is due to a difference in the spectrum properties of the Hamiltonian $H_q = -(1/2)\partial_x^2 + q\delta(x)$. In fact, if $q \geq 0$, then H_q has no eigenvalues and if $q < 0$, H_q has precisely one negative, simple eigenvalue, see Section 2 below. Hence for the attractive case, we need to analyze H_q by taking into account of the presence of the eigenfunction.

In proving Theorem 1.1, the one of key points is choice of the asymptotic profile u_{ap} . To explain how to choose u_{ap} , we employ the argument due to Ozawa [17]. We first rewrite the final state problem (1.1)-(1.2) as the integral equation of Yang-Feldman type

$$u(t) - u_{ap}(t) = i \int_t^\infty e^{-i(t-\tau)H_q} \{ \lambda |u|^2 u - (i\partial_t - \frac{1}{2}H_q)u \}(\tau) d\tau. \quad (1.5)$$

We construct the solution to (1.5) via the contraction mapping principle in a suitable Banach space. To this end, we split the right hand side of (1.5) into the following two pieces

$$\begin{aligned} u(t) - u_{ap}(t) &= i\lambda \int_t^\infty e^{-i(t-\tau)H_q} \{ |u|^2 u - |u_{ap}|^2 u_{ap} \}(\tau) d\tau \\ &\quad - i \int_t^\infty e^{-i(t-\tau)H_q} \{ (i\partial_t - \frac{1}{2}H_q)u_{ap} - \lambda |u_{ap}|^2 u_{ap} \}(\tau) d\tau. \end{aligned}$$

To apply the Banach fixed point theorem for our problem, we are required that the L^2 norm of the integrant appearing in the second term of the above equation decays faster than $t^{-5/4}$. To respond to the requirement, we take

$$u_{ap}(t, x) = t^{-1/2} \mathcal{F}_q[\phi_+](\frac{x}{t}) e^{\frac{ix^2}{2t} - \frac{i\pi}{4} + iS(t, \frac{x}{t})}.$$

Note that u_{ap} is the leading term of $e^{-itH_q}\phi_+$ with the phase modification $S(t, x/t)$. Then an elementary calculation yields

$$(i\partial_t - \frac{1}{2}H_q)u_{ap} - \lambda|u_{ap}|^2u_{ap} \sim (-\partial_t S(t, \frac{x}{t}) - \lambda t^{-1}|\mathcal{F}_q[\phi_+](\frac{x}{t})|^2)u_{ap}.$$

Therefore taking $S(t, \xi) = -\lambda|\mathcal{F}_q[\phi_+](\xi)|^2 \log t$, we obtain the desired estimate and we can apply the fixed point argument. In this step we have to impose the strong assumption on the final data. To weaken the assumption on the final data, we use a modified version of the integral equation of Yang-Feldman type which is introduced by Hayashi-Naumkin [11].

The another crucial part of this paper is the derivation of the asymptotic formula:

$$(e^{-itH_q}\phi)(x) \sim t^{-1/2}\mathcal{F}_q[\phi](\frac{x}{t})e^{\frac{ix^2}{2t} - \frac{i\pi}{4}}$$

as $t \rightarrow +\infty$ in L^p with $2 \leq p \leq \infty$, see Proposition 3.1 below for the detail. To derive this formula, we use several nontrivial identities related to the distorted Fourier transform associated to H_q . The distorted Fourier transform and its inverse transform for H_q are defined by

$$\mathcal{F}_q[\phi](\xi) = \begin{cases} \mathcal{F}[\mathcal{L}_+[\phi](x)](\xi) & \text{if } \xi \geq 0, \\ \mathcal{F}[\mathcal{L}_-[\phi](x)](\xi) & \text{if } \xi < 0, \end{cases} \quad \mathcal{F}_q^{-1}[\phi](x) = \begin{cases} \mathcal{F}_{q,+}^{-1}[\phi](x) & \text{if } x \geq 0, \\ \mathcal{F}_{q,-}^{-1}[\phi](x) & \text{if } x < 0, \end{cases}$$

for some operators \mathcal{L}_\pm and $\mathcal{F}_{q,\pm}^{-1}$, see Section 3 for the explicit forms of those operators. Especially we derive the identities

$$\begin{aligned} \mathcal{L}_+[\mathcal{F}_q^{-1}[\phi]](x) &= \mathcal{F}_{q,+}^{-1}[\phi](x), \\ \mathcal{L}_-[\mathcal{F}_q^{-1}[\phi]](x) &= \mathcal{F}_{q,-}^{-1}[\phi](x) \end{aligned}$$

for any $x \in \mathbb{R}$. Using the above identities and several properties of the distorted Fourier transform, we guarantee Proposition 3.1.

We introduce several notation and functional spaces. Let $1_+(x)$ and $1_-(x)$ be characteristic functions in x on the intervals $[0, \infty)$ and $(-\infty, 0)$. Let $\langle \xi \rangle = \sqrt{\xi^2 + 1}$ and $\langle i\partial_x \rangle = \sqrt{1 - \partial_x^2}$. For $\alpha, \beta \in \mathbb{R}$, we denote $H^{\alpha, \beta}$ the weighted Sobolev spaces

$$\begin{aligned} H^{\alpha, \beta} &= \{\phi \in \mathcal{S}'(\mathbb{R}); \|\phi\|_{H^{\alpha, \beta}} < \infty\}, \\ \|\phi\|_{H^{\alpha, \beta}} &= \|\langle x \rangle^\beta \langle i\partial_x \rangle^\alpha \phi\|_{L_x^2}. \end{aligned}$$

For $1 \leq q, r \leq \infty$, $L^q(t, \infty; L_x^r(\mathbb{R}))$ is defined as follows:

$$\begin{aligned} L^q(t, \infty; L_x^r(\mathbb{R})) &= \{u \in \mathcal{S}'(\mathbb{R}^2); \|u\|_{L^q(t, \infty; L_x^r)} < \infty\}, \\ \|u\|_{L^q(t, \infty; L_x^r)} &= \left(\int_t^\infty \|u(\tau)\|_{L_x^r}^q d\tau \right)^{1/q}. \end{aligned}$$

We use the notation $A \lesssim B$ to denote the estimate $A \leq CB$ where C is a positive constant.

In Section 2, we study several properties of the Hamiltonian H_q . Section 3 is devoted to introducing the distorted Fourier transform associated to H_q and giving an asymptotic formula as $t \rightarrow \infty$ for the unitary group e^{-itH_q} . In Section 4, we conclude the proof

of Theorem 1.1 via the contraction mapping principle. In Appendix, we prove the key identities in Lemma 3.1.

2. LINEAR ESTIMATES

We consider the Hamiltonian associated to linear Schrödinger equation with repulsive delta potential:

$$H_q = -\frac{1}{2} \frac{d^2}{dx^2} + q\delta(x) \quad \xi \in \mathbb{R}.$$

Let

$$\begin{cases} D(H_q) = \{u \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{0\}); u'(0+) - u'(0-) = 2qu(0)\}, \\ H_q u = -\frac{1}{2} \frac{d^2}{dx^2} u. \end{cases}$$

Then H_q is self-adjoint operator in $L^2(\mathbb{R})$, see [1, Theorem 3.1.1 in Chapter 1.3]. Hence the Stone theorem yields that H_q generates the L^2 -unitary group $\{e^{-itH_q}\}_{t \in \mathbb{R}}$. Furthermore, for $q \in \mathbb{R}$, the essential spectrum of H_q is purely absolutely continuous and

$$\sigma_{ess}(H_q) = \sigma_{ac}(H_q) = [0, \infty), \quad \sigma_{sc}(H_q) = \emptyset,$$

see [1, Theorem 3.1.4 in Chapter 1.3].

Remark 2.1. In [1, Theorem 3.1.4 in Chapter 1.3], it was shown that if $q \geq 0$, H_q has no eigenvalues and if $q < 0$, H_q has precisely one negative, simple eigenvalue. This fact is the difference between the repulsive and attractive cases.

Henceforth we consider the case $q \geq 0$. To obtain the explicit formula for the unitary group $\{e^{-itH_q}\}_{t \in \mathbb{R}}$, we introduce the Jost function associated to H_q .

Let $f_{\pm} = f_{\pm}(x, \xi)$ be the unique solutions to the equation

$$H_q f = \frac{1}{2} \xi^2 f$$

satisfying the asymptotic conditions

$$f_{\pm}(x, \xi) - e^{\pm i x \xi} \rightarrow 0, \quad \text{as } x \rightarrow \pm \infty.$$

Then the elementary calculation yields

$$\begin{aligned} f_+(x, \xi) &= \begin{cases} e^{i x \xi} & \text{if } x \geq 0, \\ \frac{1}{t_q(\xi)} e^{i x \xi} + \frac{r_q(\xi)}{t_q(\xi)} e^{-i x \xi} & \text{if } x < 0, \end{cases} \\ f_-(x, \xi) &= \begin{cases} \frac{1}{t_q(\xi)} e^{-i x \xi} + \frac{r_q(\xi)}{t_q(\xi)} e^{i x \xi} & \text{if } x \geq 0, \\ e^{-i x \xi} & \text{if } x < 0, \end{cases} \end{aligned}$$

where t_q and r_q are the transmission and reflection coefficients:

$$t_q(\xi) = \frac{i\xi}{i\xi - q}, \quad r_q(\xi) = \frac{q}{i\xi - q}. \quad (2.1)$$

To obtain the the following representation formula for e^{-itH_q} , we introduce the following two operators:

$$\mathcal{L}_+[\phi](x) = \phi(x) - q1_-(x)e^{qx} \int_x^{-x} e^{q|y|} \phi(y) dy, \quad (2.2)$$

$$\mathcal{L}_-[\phi](x) = \phi(x) - q1_+(x)e^{-qx} \int_{-x}^x e^{q|y|} \phi(y) dy, \quad (2.3)$$

Proposition 2.1. *Let $\phi \in L^2(\mathbb{R})$. Then we have*

$$(e^{-itH_q}\phi)(x) = \begin{cases} \frac{e^{-i\pi/4}}{\sqrt{2\pi}} t^{-1/2} \int_{-\infty}^{\infty} e^{\frac{i(x-y)^2}{2t}} \mathcal{L}_+[\phi](y) dy & \text{if } x \geq 0, \\ \frac{e^{-i\pi/4}}{\sqrt{2\pi}} t^{-1/2} \int_{-\infty}^{\infty} e^{\frac{i(x-y)^2}{2t}} \mathcal{L}_-[\phi](y) dy & \text{if } x < 0, \end{cases} \quad (2.4)$$

To prove this proposition, we show the following lemma.

Lemma 2.1. *Let $\phi \in L^2(\mathbb{R})$. Then we have*

$$\begin{aligned} & (e^{-itH_q}\phi)(x) \\ &= \begin{cases} \mathcal{F}^{-1}[e^{-it\xi^2/2} \{ \mathcal{F}[\phi](\xi) + r_q(\xi) \mathcal{F}[1_+\phi](-\xi) + r_q(\xi) \mathcal{F}[1_-\phi](\xi) \}](x) & \text{if } x \geq 0, \\ \mathcal{F}^{-1}[e^{-it\xi^2/2} \{ \mathcal{F}[\phi](\xi) + \overline{r_q(\xi)} \mathcal{F}[1_-\phi](-\xi) + \overline{r_q(\xi)} \mathcal{F}[1_+\phi](\xi) \}](x) & \text{if } x < 0, \end{cases} \end{aligned} \quad (2.5)$$

Proof of Lemma 2.1. The explicit representation for the spectral projection implies

$$\begin{aligned} & (e^{-itH_q}\phi)(x) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_0^{\infty} e^{-\frac{i}{2}t\xi^2} |t_q(\xi)|^2 \right. \\ & \quad \times \left(f_+(x, \xi) \overline{f_+(y, \xi)} + f_-(x, \xi) \overline{f_-(y, \xi)} \right) d\xi \Big\} \phi(y) dy. \end{aligned} \quad (2.6)$$

We first consider the case $x \geq 0$. In this case, we have

$$\begin{aligned} & (e^{-itH_q}\phi)(x) \\ &= \frac{1}{2\pi} \int_0^{\infty} e^{-\frac{i}{2}t\xi^2} |t_q(\xi)|^2 f_+(x, \xi) \left(\int_0^{\infty} \overline{f_+(y, \xi)} \phi(y) dy \right) d\xi \\ & \quad + \frac{1}{2\pi} \int_0^{\infty} e^{-\frac{i}{2}t\xi^2} |t_q(\xi)|^2 f_+(x, \xi) \left(\int_{-\infty}^0 \overline{f_+(y, \xi)} \phi(y) dy \right) d\xi \\ & \quad + \frac{1}{2\pi} \int_0^{\infty} e^{-\frac{i}{2}t\xi^2} |t_q(\xi)|^2 f_-(x, \xi) \left(\int_0^{\infty} \overline{f_-(y, \xi)} \phi(y) dy \right) d\xi \\ & \quad + \frac{1}{2\pi} \int_0^{\infty} e^{-\frac{i}{2}t\xi^2} |t_q(\xi)|^2 f_-(x, \xi) \left(\int_{-\infty}^0 \overline{f_-(y, \xi)} \phi(y) dy \right) d\xi \\ &\equiv F_1(t, x) + F_2(t, x) + F_3(t, x) + F_4(t, x). \end{aligned}$$

From the definition of f_{\pm} , we have

$$\begin{aligned}
F_1(t, x) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{ix\xi - \frac{i}{2}t\xi^2} |t_q(\xi)|^2 \mathcal{F}[1_+\phi](\xi) d\xi, \\
F_2(t, x) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{ix\xi - \frac{i}{2}t\xi^2} t_q(\xi) \mathcal{F}[1_-\phi](\xi) d\xi \\
&\quad + \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{ix\xi - \frac{i}{2}t\xi^2} t_q(\xi) \overline{r_q(\xi)} \mathcal{F}[1_-\phi](-\xi) d\xi \\
&\equiv F_{21}(t, x) + F_{22}(t, x), \\
F_3(t, x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{ix\xi - \frac{i}{2}t\xi^2} \mathcal{F}[1_+\phi](\xi) d\xi \\
&\quad + \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{ix\xi - \frac{i}{2}t\xi^2} r_q(\xi) \mathcal{F}[1_+\phi](-\xi) d\xi \\
&\quad + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{ix\xi - \frac{i}{2}t\xi^2} r_q(\xi) \mathcal{F}[1_+\phi](-\xi) d\xi \\
&\quad + \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{ix\xi - \frac{i}{2}t\xi^2} |r_q(\xi)|^2 \mathcal{F}[1_+\phi](\xi) d\xi \\
&\equiv F_{31}(t, x) + F_{32}(t, x) + F_{33}(t, x) + F_{34}(t, x), \\
F_4(t, x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{ix\xi - \frac{i}{2}t\xi^2} t_q(\xi) \mathcal{F}[1_-\phi](\xi) d\xi \\
&\quad + \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{ix\xi - \frac{i}{2}t\xi^2} \overline{t_q(\xi)} r_q(\xi) \mathcal{F}[1_-\phi](-\xi) d\xi \\
&\equiv F_{41}(t, x) + F_{42}(t, x).
\end{aligned}$$

Since $|t_q(\xi)|^2 + |r_q(\xi)|^2 = 1$, we have

$$F_1(t, x) + F_{31}(t, x) + F_{34}(t, x) = \mathcal{F}^{-1}[e^{-it\xi^2/2} \mathcal{F}[1_+\phi](\xi)](x).$$

The identity $t_q(\xi) = 1 + r_q(\xi)$ yields

$$\begin{aligned}
&F_{21}(t, x) + F_{41}(t, x) \\
&= \mathcal{F}^{-1}[e^{-it\xi^2/2} \mathcal{F}[1_-\phi](\xi)](x) + \mathcal{F}^{-1}[e^{-it\xi^2/2} r_q(\xi) \mathcal{F}[1_-\phi](\xi)](x).
\end{aligned}$$

Using the relation $t_q(\xi) \overline{r_q(\xi)} + \overline{t_q(\xi)} t_q(\xi) = 0$, we obtain

$$F_{22}(t, x) + F_{42}(t, x) = 0.$$

Combining the above identities with a trivial identity

$$F_{32}(t, x) + F_{33}(t, x) = \mathcal{F}^{-1}[e^{-it\xi^2/2} r_q(\xi) \mathcal{F}[1_+\phi](-\xi)](x),$$

we have (2.5) with $x \geq 0$.

Next we consider the case $x < 0$. Notice that $f_+(x, \xi) = f_-(-x, \xi)$. Then (2.2) implies

$$\begin{aligned}
&(e^{-itH_q} \phi)(x) \\
&= \frac{1}{2\pi} \int_{-\infty}^\infty \left\{ \int_0^\infty e^{-\frac{i}{2}t\xi^2} |t_q(\xi)|^2 \right.
\end{aligned}$$

$$\begin{aligned}
& \times \left(f_-(-x, \xi) \overline{f_-(-y, \xi)} + f_+(-x, \xi) \overline{f_+(-y, \xi)} \right) d\xi \Big\} \phi(y) dy \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_0^{\infty} e^{-\frac{i}{2}t\xi^2} |t_q(\xi)|^2 \right. \\
&\quad \times \left(f_-(-x, \xi) \overline{f_-(y, \xi)} + f_+(-x, \xi) \overline{f_+(y, \xi)} \right) d\xi \Big\} \phi(-y) dy \\
&= (e^{-itH_q} \phi(-\cdot))(-x).
\end{aligned}$$

Hence the identity (2.5) with $x < 0$ follows from (2.5) with $x \geq 0$. This completes the proof of Lemma 2.1. \square

Proof of Proposition 2.1. We consider the case $x \geq 0$ only because the identity (2.4) with the case $x < 0$ follows from (2.4) with the case $x \geq 0$ through the relation $(e^{-itH_q} \phi)(x) = (e^{-itH_q} \phi(-\cdot))(-x)$. By Lemma 2.1, we obtain

$$\begin{aligned}
& (e^{-itH_q} \phi)(x) \\
&= \left(\frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}[e^{-it\xi^2/2}] * \phi \right)(x) + \frac{1}{2\pi} (\mathcal{F}^{-1}[e^{-it\xi^2/2}] * \mathcal{F}^{-1}[r_q] * (1_+ \phi)(-\cdot))(x) \\
&\quad + \frac{1}{2\pi} (\mathcal{F}^{-1}[e^{-it\xi^2/2}] * \mathcal{F}^{-1}[r_q] * (1_- \phi))(x).
\end{aligned}$$

Since $\mathcal{F}^{-1}[r_q](x) = -\sqrt{2\pi}q1_-(x)e^{qx}$, we obtain

$$\begin{aligned}
(\mathcal{F}^{-1}[r_q] * (1_+ \phi)(-\cdot))(x) &= -\sqrt{2\pi}q1_-(x)e^{qx} \int_0^{-x} e^{qy} \phi(y) dy, \\
(\mathcal{F}^{-1}[r_q] * 1_- \phi)(x) &= -\sqrt{2\pi}q1_-(x)e^{qx} \int_x^0 e^{-qy} \phi(y) dy.
\end{aligned}$$

Combining the above identities with the well-known identity

$$\mathcal{F}^{-1}[e^{-it\xi^2/2}](x) = e^{-i\pi/4} t^{-1/2} e^{\frac{ix^2}{2t}},$$

we obtain (2.4). \square

Next lemma is concerned with the Strichartz estimates for the group e^{-itH_q} which plays an important role to construct the solution to (1.1) in the finite interval $[0, T)$ (see Lemma 2.3 below) and the infinite interval $[T, \infty)$ (see Section 4).

Lemma 2.2. *Let $2/q_j + 1/r_j = 1/2$, $4 \leq q_j \leq \infty$ and $j = 1, 2$. For any (possibly unbounded) interval I and for any $s \in \bar{I}$, we have*

$$\left\| \int_s^t e^{-i(t-\tau)H_q} F(\tau) d\tau \right\|_{L^{q_1}(I; L_x^{r_1})} \lesssim \|F\|_{L^{q'_2}(I; L_x^{r'_2})}, \quad (2.7)$$

where p' is the Hölder conjugate exponent of p , and the implicit constant depends only on p .

Proof of Lemma 2.2. Although Lemma 2.2 is proved in [13, Proposition 2.2], we give the proof of this lemma for the reader's convenience. From the L^2 unitarity of e^{-itH_q} , we

have

$$\|e^{-itH_q}\phi\|_{L_x^2(\mathbb{R})} = \|\phi\|_{L_x^2(\mathbb{R})}.$$

From Proposition 2.1 (2.4) and the fact that $\mathcal{L}_\pm : L^1 \rightarrow L^1$ is bounded, we have

$$\begin{aligned} \|e^{-itH_q}\phi\|_{L_x^\infty(\mathbb{R})} &\lesssim t^{-1/2}(\|\mathcal{L}_+[\phi]\|_{L_x^1} + \|\mathcal{L}_-[\phi]\|_{L_x^1}) \\ &\lesssim t^{-1/2}\|\phi\|_{L_x^1}. \end{aligned}$$

Combining the above two boundness for e^{-itH_q} with TT^* argument (see Yajima [20] and Keel-Tao[15] for instance) we obtain (2.7). \square

We next state the lemma concerning the global well-posedness for (1.1) in $L^2(\mathbb{R})$.

Lemma 2.3. *For any $\phi \in L^2(\mathbb{R})$, there exists a unique solution $u \in C(\mathbb{R}; L^2(\mathbb{R})) \cap L_{loc}^q(\mathbb{R}; L_x^r(\mathbb{R}))$ to (1.1) with $u(0, x) = \phi(x)$, where (q, r) satisfies $2/q + 1/r = 1/2$ and $4 \leq q \leq \infty$.*

Proof of Lemma 2.3. The proof follows from the the Strichartz estimate (Lemma 2.2 (2.7) with $I = [0, T)$ and $s = 0$) and the fixed point argument, see Tsutsumi [18] for the detail. \square

3. DISTORTED FOURIER TRANSFORM

In this section, we introduce a distorted Fourier transform associated to H_q .

Let us review that the usual Fourier transform and its inverse transform are defined by

$$\begin{aligned} \mathcal{F}[\phi](\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-iy\xi} \phi(y) dy, \\ \mathcal{F}^{-1}[\phi](x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ix\xi} \phi(\xi) d\xi. \end{aligned}$$

Let t_q and r_q are defined by (2.1). We introduce

$$\Psi(x, \xi) = \begin{cases} t_q(\xi) f_+(x, \xi) & \text{if } \xi \geq 0, \\ t_q(-\xi) f_-(x, -\xi) & \text{if } \xi < 0. \end{cases}$$

Then the distorted Fourier transform and its inverse transform for H_q are defined by

$$\begin{aligned} \mathcal{F}_q[\phi](\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi(y, -\xi) \phi(y) dy, \\ \mathcal{F}_q^{-1}[\phi](x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \overline{\Psi(x, -\xi)} \phi(\xi) d\xi. \end{aligned}$$

By the definition, the relations between the usual and distorted Fourier transforms are as follows:

$$\mathcal{F}_q[\phi](\xi) = \begin{cases} \mathcal{F}[\phi](\xi) + r_q(\xi) \mathcal{F}[1_+\phi](-\xi) + r_q(\xi) \mathcal{F}[1_-\phi](\xi) & \text{if } \xi \geq 0, \\ \mathcal{F}[\phi](\xi) + \overline{r_q(\xi)} \mathcal{F}[1_-\phi](-\xi) + \overline{r_q(\xi)} \mathcal{F}[1_+\phi](\xi) & \text{if } \xi < 0, \end{cases} \quad (3.1)$$

$$\mathcal{F}_q^{-1}[\phi](x) = \begin{cases} \mathcal{F}^{-1}[\phi](x) + \mathcal{F}^{-1}[1_+ \overline{r_q} \phi](-x) + \mathcal{F}^{-1}[1_- r_q \phi](x) & \text{if } x \geq 0, \\ \mathcal{F}^{-1}[\phi](x) + \mathcal{F}^{-1}[1_- r_q \phi](-x) + \mathcal{F}^{-1}[1_+ \overline{r_q} \phi](x) & \text{if } x < 0, \end{cases} \quad (3.2)$$

Furthermore, we have the another representation for the distorted Fourier transforms:

$$\begin{aligned} \mathcal{F}_q[\phi](\xi) &= \begin{cases} \mathcal{F}[\phi(x) - q1_-(x)e^{qx} \int_x^{-x} e^{q|y|} \phi(y) dy](\xi) & \text{if } \xi \geq 0, \\ \mathcal{F}[\phi(x) - q1_+(x)e^{-qx} \int_{-x}^x e^{q|y|} \phi(y) dy](\xi) & \text{if } \xi < 0, \end{cases} \\ &= \begin{cases} \mathcal{F}[\mathcal{L}_+[\phi](x)](\xi) & \text{if } \xi \geq 0, \\ \mathcal{F}[\mathcal{L}_-[\phi](x)](\xi) & \text{if } \xi < 0, \end{cases} \end{aligned} \quad (3.3)$$

$$\mathcal{F}_q^{-1}[\phi](x) = \begin{cases} \mathcal{F}^{-1}[\phi](x) - qe^{qx} \int_{-\infty}^{-x} e^{qy} \mathcal{F}^{-1}[1_+ \phi](y) dy \\ \quad - qe^{qx} \int_x^{+\infty} e^{-qy} \mathcal{F}^{-1}[1_- \phi](y) dy & \text{if } x \geq 0, \\ \mathcal{F}^{-1}[\phi](x) - qe^{-qx} \int_{-\infty}^x e^{qy} \mathcal{F}^{-1}[1_+ \phi](y) dy \\ \quad - qe^{-qx} \int_x^{+\infty} e^{-qy} \mathcal{F}^{-1}[1_- \phi](y) dy & \text{if } x < 0. \end{cases} \quad (3.4)$$

From (3.3) and (3.4), we see that

$$\mathcal{F}_q \mathcal{F}_q^{-1} = \mathcal{F}_q^{-1} \mathcal{F}_q = I \quad \text{in } L^2(\mathbb{R})$$

and

$$\int_{\mathbb{R}} \mathcal{F}_q[\phi](x) \overline{\varphi(x)} dx = \int_{\mathbb{R}} \phi(x) \overline{\mathcal{F}_q^{-1}[\varphi](x)} dx$$

for any $\phi, \varphi \in L^2(\mathbb{R})$. From the above identities, we have the L^2 isometry of \mathcal{F}_q and \mathcal{F}_q^{-1} :

$$\|\mathcal{F}_q[\phi]\|_{L_\xi^2} = \|\phi\|_{L_x^2}, \quad \|\mathcal{F}_q^{-1}[\varphi]\|_{L_x^2} = \|\varphi\|_{L_\xi^2}.$$

Remark 3.1. We can express the unitary group e^{-itH_q} in terms of the distorted Fourier transform. Indeed, we obtain

$$(e^{-itH_q} \phi)(x) = \overline{\mathcal{F}_q^{-1} \left[e^{\frac{i}{2} t \xi^2} \mathcal{F}_q [\overline{\phi}] (\xi) \right]} (x)$$

for any $\phi \in L^2(\mathbb{R})$.

Next proposition is concerned with the asymptotic behavior of the unitary group e^{-itH_q} .

Proposition 3.1. *Let $\phi \in \mathcal{S}'(\mathbb{R})$ satisfy $\phi, \partial_\xi \phi \in L_\xi^2$ and $\phi(0) = 0$. Let $2 \leq p \leq \infty$ and β satisfy $0 \leq \beta \leq 1$ for $p = 2$ and $0 \leq \beta < (p+2)/(2p)$ for $2 < p \leq \infty$. Then we have*

$$(e^{-itH_q} \mathcal{F}_q^{-1} \phi)(x) = t^{-1/2} \phi\left(\frac{x}{t}\right) e^{\frac{ix^2}{2t} - \frac{i\pi}{4}} + R(t, x),$$

where R satisfies

$$\|R(t)\|_{L_x^p} \lesssim t^{-1/2+1/p-\beta/2}(\|\phi\|_{L_\xi^2} + \|\partial_\xi \phi\|_{L_\xi^2}) \quad (3.5)$$

for $t > 0$.

To prove Proposition 3.1, we show the following identities.

Lemma 3.1. *Let \mathcal{L}_\pm be defined by (2.2) and (2.3). Then we have*

$$\mathcal{L}_+[\mathcal{F}_q^{-1}[\phi]](y) = \mathcal{F}_{q,+}^{-1}[\phi](y), \quad (3.6)$$

$$\mathcal{L}_-[\mathcal{F}_q^{-1}[\phi]](y) = \mathcal{F}_{q,-}^{-1}[\phi](y) \quad (3.7)$$

for any $y \in \mathbb{R}$, where $\mathcal{F}_{q,\pm}^{-1}$ are defined by

$$\begin{aligned} \mathcal{F}_{q,+}^{-1}[\phi](y) &= \mathcal{F}^{-1}[\phi](x) - qe^{qx} \int_{-\infty}^{-x} e^{qy} \mathcal{F}^{-1}[1_+\phi](y) dy \\ &\quad - qe^{qx} \int_x^{+\infty} e^{-qy} \mathcal{F}^{-1}[1_-\phi](y) dy, \\ \mathcal{F}_{q,-}^{-1}[\phi](y) &= \mathcal{F}^{-1}[\phi](x) - qe^{-qx} \int_{-\infty}^x e^{qy} \mathcal{F}^{-1}[1_+\phi](y) dy \\ &\quad - qe^{-qx} \int_{-x}^{+\infty} e^{-qy} \mathcal{F}^{-1}[1_-\phi](y) dy. \end{aligned}$$

Remark 3.2. The operators $\mathcal{F}_{q,\pm}^{-1}$ are appeared in the representation formula (3.4) for \mathcal{F}_q^{-1} .

We shall prove Lemma 3.1 in Appendix below.

Proof of Proposition 3.1. We consider the case $x \geq 0$ only. From Proposition 2.1 (2.4) we have

$$\begin{aligned} (e^{-itH_q} \mathcal{F}_q^{-1} \phi)(x) &= \frac{e^{-i\pi/4}}{\sqrt{2\pi}} t^{-1/2} e^{\frac{ix^2}{2t}} \int_{-\infty}^{+\infty} e^{-i\frac{x}{t}y} e^{\frac{iy^2}{2t}} \mathcal{L}_+[\mathcal{F}_q^{-1}[\phi]](y) dy \\ &= \frac{e^{-i\pi/4}}{\sqrt{2\pi}} t^{-1/2} e^{\frac{ix^2}{2t}} \int_{-\infty}^{+\infty} e^{-i\frac{x}{t}y} \mathcal{L}_+[\mathcal{F}_q^{-1}[\phi]](y) dy \\ &\quad + \frac{e^{-i\pi/4}}{\sqrt{2\pi}} t^{-1/2} e^{\frac{ix^2}{2t}} \int_{-\infty}^{+\infty} e^{-i\frac{x}{t}y} (e^{\frac{iy^2}{2t}} - 1) \mathcal{L}_+[\mathcal{F}_q^{-1}[\phi]](y) dy \\ &\equiv L(t, x) + R(t, x). \end{aligned}$$

From the representation formula (3.3) for \mathcal{F}_q , we see

$$L(t, x) = t^{-1/2} \phi\left(\frac{x}{t}\right) e^{\frac{ix^2}{2t} - \frac{i\pi}{4}}.$$

For R , we evaluate as

$$\|R(t)\|_{L_x^p(0,\infty)} \lesssim t^{-1/2+1/p} \|(e^{\frac{iy^2}{2t}} - 1) \mathcal{L}_+[\mathcal{F}_q^{-1}[\phi]](y)\|_{L_y^{p'}(\mathbb{R})}$$

$$\begin{aligned}
&\lesssim t^{-1/2+1/p-\beta/2} \| |y|^\beta \mathcal{L}_+[\mathcal{F}_q^{-1}[\phi]](y) \|_{L_y^{p'}(\mathbb{R})} \\
&\lesssim t^{-1/2+1/p-\beta/2} \| \langle y \rangle^{\beta+\gamma} \mathcal{L}_+[\mathcal{F}_q^{-1}[\phi]](y) \|_{L_y^2(\mathbb{R})},
\end{aligned}$$

where $2 \leq p \leq \infty$, $0 \leq \beta \leq 1$ and γ satisfies $\gamma = 0$ for $p = 2$ and $\gamma > (p-2)/(2p)$ for $2 < p \leq \infty$. We choose β so that $\beta + \gamma = 1$. Using Lemma 3.1 and the representation (3.2) for $\mathcal{F}_{q,+}^{-1}$, we obtain

$$\begin{aligned}
&\| \langle y \rangle \mathcal{L}_+[\phi](y) \|_{L_y^2(\mathbb{R})} \\
&\leq \| \mathcal{F}^{-1}[\phi](y) \|_{L_y^2(\mathbb{R})} + \| y \mathcal{F}^{-1}[\phi](y) \|_{L_y^2(\mathbb{R})} \\
&\quad + \| \mathcal{F}^{-1}[1_+ \bar{r}_q \phi](-y) \|_{L_y^2(\mathbb{R})} + \| y \mathcal{F}^{-1}[1_+ \bar{r}_q \phi](-y) \|_{L_y^2(\mathbb{R})} \\
&\quad + \| \mathcal{F}^{-1}[1_- r_q \phi](y) \|_{L_y^2(\mathbb{R})} + \| y \mathcal{F}^{-1}[1_- r_q \phi](y) \|_{L_y^2(\mathbb{R})} \\
&\equiv I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}$$

Since $|r_q(\xi)| + |r'_q(\xi)| \lesssim 1$, we see that

$$I_1 + I_3 + I_5 \lesssim \| \mathcal{F}_q[\phi] \|_{L_\xi^2}.$$

Notice that if $\varphi(0) = 0$, then $y \mathcal{F}^{-1}[1_\pm \varphi](y) = i \mathcal{F}^{-1}[1_\pm \partial_\xi \varphi](y)$. Hence we have

$$\begin{aligned}
I_2 &= \| \mathcal{F}^{-1}[\partial_\xi \phi] \|_{L_y^2(\mathbb{R})} \lesssim \| \partial_\xi \phi \|_{L_\xi^2}, \\
I_4 &= \| \mathcal{F}^{-1}[1_+ \partial_\xi (\bar{r}_q \phi)] \|_{L_y^2(\mathbb{R})} \lesssim \| \phi \|_{L_\xi^2} + \| \partial_\xi \phi \|_{L_\xi^2}, \\
I_6 &= \| \mathcal{F}^{-1}[1_- \partial_\xi (r_q \phi)] \|_{L_y^2(\mathbb{R})} \lesssim \| \phi \|_{L_\xi^2} + \| \partial_\xi \phi \|_{L_\xi^2}.
\end{aligned}$$

Therefore we obtain (3.5) for the case $x \geq 0$. By a similar way as above, we obtain (3.5) for the case $x < 0$ which completes the proof of Proposition 3.1. \square

4. CONSTRUCTION OF MODIFIED WAVE OPERATORS

In this section we prove Theorem 1.1. We first rewrite (1.1) as the integral equation. Using Proposition 3.1, we have

$$u(t) - u_{ap}(t) = u(t) - e^{-itH_q} \mathcal{F}_q^{-1} w + R_1(t), \quad (4.1)$$

where

$$w(t, \xi) = \mathcal{F}_q[\phi_+](\xi) e^{-i\lambda |\mathcal{F}_q[\phi_+](\xi)|^2 \log t}, \quad (4.2)$$

and R_1 satisfies

$$\begin{aligned}
&\| R_1(t) \|_{L^\infty(t, \infty; L_x^2)} \\
&\lesssim t^{-\beta/2} (\| \mathcal{F}_q[\phi_+] \|_{L_\xi^2} + \| \partial_\xi \mathcal{F}_q[\phi_+] \|_{L_\xi^2}) (1 + \| \mathcal{F}_q[\phi_+] \|_{L_\xi^2} + \| \partial_\xi \mathcal{F}_q[\phi_+] \|_{L_\xi^2})^2
\end{aligned}$$

for $0 \leq \beta \leq 1$, where we used the fact that $\mathcal{F}_q[\phi_+](0) = 0$. From the representation formula (3.1) for \mathcal{F}_q , we see that

$$\| \mathcal{F}_q[\phi_+] \|_{L_\xi^2} = \| \phi_+ \|_{L_x^2}, \quad \| \partial_\xi \mathcal{F}_q[\phi_+] \|_{L_\xi^2} \lesssim \| \phi_+ \|_{H_x^{0,1}}.$$

Hence we have

$$\| R_1(t) \|_{L^\infty(t, \infty; L_x^2)} \lesssim t^{-\beta/2} \| \phi_+ \|_{H_x^{0,1}} (1 + \| \phi_+ \|_{H_x^{0,1}})^2. \quad (4.3)$$

By a similar way we obtain

$$\|R_1(t)\|_{L^4(t,\infty;L_x^\infty)} \lesssim t^{-\gamma/2-1/4}\|\phi_+\|_{H_x^{0,1}}(1+\|\phi_+\|_{H_x^{0,1}})^2, \quad (4.4)$$

where $0 \leq \gamma < 1/2$. Eq. (4.1) is rewritten as follows:

$$u(t) - u_{ap}(t) = e^{-itH_q} \mathcal{F}_q^{-1} [\mathcal{F}_q e^{itH_q} u - w] + R_1(t). \quad (4.5)$$

From (1.1) and (4.2), we obtain

$$i\partial_t(\mathcal{F}_q e^{itH_q} u) = \lambda \mathcal{F}_q e^{itH_q} |u|^2 u, \quad (4.6)$$

$$i\partial_t w = \lambda t^{-1} |w|^2 w. \quad (4.7)$$

Subtracting (4.7) from (4.6), we have

$$i\partial_t(\mathcal{F}_q e^{itH_q} u - w) = \lambda \mathcal{F}_q e^{itH_q} |u|^2 u - \lambda t^{-1} |w|^2 w. \quad (4.8)$$

Proposition 3.1 yields

$$e^{-itH_q} \mathcal{F}_q^{-1} [\lambda t^{-1} |w|^2 w] = \lambda |u_{ap}|^2 u_{ap} + R_2(t, x), \quad (4.9)$$

where R_2 satisfies

$$\|R_2\|_{L^1(t,\infty;L_x^2)} \lesssim t^{-\beta/2} \|\phi_+\|_{H_x^{0,1}} (1 + \|\phi_+\|_{H_x^{0,1}})^4, \quad (4.10)$$

where $0 \leq \beta \leq 1$. Substituting (4.9) into (4.8), we obtain

$$i\partial_t(\mathcal{F}_q e^{itH_q} u - w) = \lambda \mathcal{F}_q e^{itH_q} [|u|^2 u - |u_{ap}|^2 u_{ap}] - \mathcal{F}_q e^{itH_q} R_2.$$

Integrating the above equation in t , we have

$$\begin{aligned} u(t) - e^{-itH_q} \mathcal{F}_q^{-1} w &= i\lambda \int_t^{+\infty} e^{-i(t-\tau)H_q} (|u|^2 u - |u_{ap}|^2 u_{ap})(\tau) d\tau \\ &\quad - i \int_t^{+\infty} e^{-i(t-\tau)H_q} R_2(\tau) d\tau. \end{aligned} \quad (4.11)$$

Combining (4.1) with (4.11), we obtain the following integral equation:

$$\begin{aligned} u(t) - u_{ap}(t) &= i\lambda \int_t^{+\infty} e^{-i(t-\tau)H_q} [|u|^2 u - |u_{ap}|^2 u_{ap}](\tau) d\tau \\ &\quad + R_1(t) - i \int_t^{+\infty} e^{-i(t-\tau)H_q} R_2(\tau) d\tau. \end{aligned} \quad (4.12)$$

To show the existence of u satisfying (4.12), we shall prove that the map

$$\begin{aligned} \Phi[u](t) &= u_{ap}(t) + i\lambda \int_t^{+\infty} e^{-i(t-\tau)H_q} [|u|^2 u - |u_{ap}|^2 u_{ap}](\tau) d\tau \\ &\quad + R_1(t) - i \int_t^{+\infty} e^{-i(t-\tau)H_q} R_2(\tau) d\tau \end{aligned}$$

is a contraction on

$$\begin{aligned} \mathbf{X}_{r,T} &= \{u \in C([T, \infty); L^2(\mathbb{R})) \cap L_{loc}^4(T, \infty; L^\infty(\mathbb{R})); \|u - u_{ap}\|_{\mathbf{X}_T} \leq r\}, \\ \|v\|_{\mathbf{X}_T} &= \sup_{t \geq T} t^\alpha (\|v\|_{L^\infty(t, \infty; L_x^2)} + \|v\|_{L^4(t, \infty; L_x^\infty)}) \end{aligned}$$

for some $r > 0$ and $T > 0$.

Let $v(t) = u(t) - u_{ap}(t)$ and $\|\phi_+\|_{H^{0,1}} \leq r$. Then for any $v \in \mathbf{X}_T$ satisfying $\|v\|_{\mathbf{X}_T} \leq r$, we obtain

$$\begin{aligned} & \Phi[u](t) - u_{ap}(t) \\ &= i\lambda \int_t^{+\infty} e^{-i(t-\tau)H_q} [|v|^2 v + 2u_{ap}|v|^2 + \bar{u}_{ap}v^2 + 2|u_{ap}|^2 v + u_{ap}^2 \bar{v}](\tau) d\tau \\ & \quad + R_1(t, x) - i \int_t^{+\infty} e^{-i(t-\tau)H_q} R_2(\tau) d\tau. \end{aligned}$$

Strichartz estimate (Lemma 2.2 (2.7) with $I = [t, \infty)$ and $s = \infty$) implies

$$\begin{aligned} & \|\Phi[u] - u_{ap}\|_{L^\infty(t, \infty; L_x^2)} + \|\Phi[u] - u_{ap}\|_{L^4(t, \infty; L_x^\infty)} \\ & \lesssim (\|F_1\|_{L^{4/3}(t, \infty; L_x^1)} + \|F_2\|_{L^1(t, \infty; L_x^2)} + \|F_3\|_{L^1(t, \infty; L_x^2)}) \\ & \quad + (\|R_1\|_{L^\infty(t, \infty; L_x^2)} + \|R_1\|_{L^4(t, \infty; L_x^\infty)} + \|R_2\|_{L^1(t, \infty; L_x^2)}), \end{aligned} \quad (4.13)$$

where $F_1 = |v|^2 v$, $F_2 = 2u_{ap}|v|^2 + \bar{u}_{ap}v^2$ and $F_3 = 2|u_{ap}|^2 v + u_{ap}^2 \bar{v}$. By the Hölder inequality,

$$\begin{aligned} \|F_1\|_{L^{4/3}(t, \infty; L_x^1)} & \leq \| |v|^2 \|_{L_x^2} \|v\|_{L_x^\infty} \|v\|_{L^{4/3}(t, \infty)} \\ & \lesssim r^2 \|t^{-2\alpha}\|_{L_x^\infty} \|v\|_{L^{4/3}(t, \infty)} \\ & \lesssim r^2 \|t^{-2\alpha}\|_{L^2(t, \infty)} \|v\|_{L^4(t, \infty; L_x^\infty)} \\ & \lesssim t^{-3\alpha+1/2} r^3, \\ \|F_2\|_{L^1(t, \infty; L_x^2)} & \leq \| |u_{ap}| \|_{L_x^\infty} \|v\|_{L_x^2} \|v\|_{L_x^\infty} \|v\|_{L^1(t, \infty)} \\ & \lesssim r^2 \|t^{-\alpha-1/2}\|_{L_x^\infty} \|v\|_{L^1(t, \infty)} \\ & \lesssim r^2 \|t^{-\alpha-1/2}\|_{L^{4/3}(t, \infty)} \|v\|_{L^4(t, \infty; L_x^\infty)} \\ & \lesssim t^{-2\alpha+1/4} r^3, \\ \|F_3\|_{L^1(t, \infty; L_x^2)} & \leq \| |u_{ap}|^2 \|_{L_x^\infty} \|v\|_{L_x^2} \|v\|_{L^1(t, \infty)} \\ & \lesssim r^3 \|t^{-\alpha-1}\|_{L^1(t, \infty)} \\ & \lesssim t^{-\alpha} r^3. \end{aligned}$$

Substituting above three inequalities, (4.3), (4.4) and (4.10) into (4.13), we have

$$\begin{aligned} & \|\Phi[u] - u_{ap}\|_{\mathbf{X}_T} \\ & \leq C(T^{-2\alpha+1/2} + 1)r^3 + C(T^{\alpha-\beta/2} + T^{\alpha-\gamma/2-1/4})r(1+r^4). \end{aligned}$$

Choosing α, β and γ so that $1/4 < \alpha < 1/2$, $2\alpha < \beta < 1$ and $2(\alpha - 1/4) < \gamma < 1/2$, and taking T large enough and r sufficiently small, we guarantee that Φ is the map onto $\mathbf{X}_{r,T}$. By a similar way we can conclude that Φ is the contraction map on $\mathbf{X}_{r,T}$. Therefore Banach fixed point theorem yields that Φ has a unique fixed point in $\mathbf{X}_{r,T}$ which is the solution to the final state problem (1.1)-(1.2).

Next, we show that the solution to (1.1) with finite \mathbf{X}_T norm is unique. Let u and v be two solutions satisfying $\|u\|_{\mathbf{X}_T} < \infty$ and $\|v\|_{\mathbf{X}_T} < \infty$. We put $t_1 = \inf\{t \in [T, \infty); u(s) = v(s) \text{ for any } s \in [t, \infty)\}$ and $R = \max\{\|u\|_{\mathbf{X}_T}, \|v\|_{\mathbf{X}_T}\}$. If $t_1 = T$, then $u(t) = v(t)$ on $[T, \infty)$ which is desired result. If $T < t_1$, as in (4.13) by the Strichartz inequality (Lemma

2.2), we have

$$\|u - v\|_{L^4(t_0, t_1, L_x^\infty)} \leq CR^2(t_0^{1-4\alpha} - t_1^{1-4\alpha})^{1/2} \|u - v\|_{L^4(t_0, t_1, L_x^\infty)},$$

for $t_0 \in [T, t_1)$. Since $-2\alpha + 1/2 < 0$, we can choose $t_0 \in [T, t_1)$ so that $CR^2(t_0^{1-4\alpha} - t_1^{1-4\alpha})^{1/2} < 1$. Then $\|u - v\|_{L^4(t_0, t_1, L_x^\infty)} \leq 0$ which implies that $u(t) \equiv v(t)$ on $[t_0, t_1]$. This contradicts the assumption of t_1 . Hence $u(t) = v(t)$ on $[T, \infty)$.

From (1.1), we obtain

$$u(t) = e^{-i(t-T)H_q}u(T) - i\lambda \int_T^t e^{-i\tau H_q}|u|^2u(\tau)d\tau. \quad (4.14)$$

Since $u(T) \in L_x^2(\mathbb{R})$, it follows from Lemma 2.3 that (4.14) has a unique global solution in $C(\mathbb{R}; L_x^2(\mathbb{R})) \cap L_{loc}^4(\mathbb{R}; L_x^\infty(\mathbb{R}))$. Therefore the solution u of (1.1) extends to all times. This completes the proof of Theorem 1.1. \square

5. APPENDIX

In this appendix, we give a proof of Lemma 3.1. Since the case $y \geq 0$ is trivial, we consider the case $y < 0$ only. By definition, we have

$$\begin{aligned} \mathcal{L}_+[\mathcal{F}_q^{-1}[\phi]](y) \\ = \mathcal{F}_q^{-1}[\phi](y) - qe^{qy} \int_y^0 e^{-qz} \mathcal{F}_q^{-1}[\phi](z)dz - qe^{qy} \int_0^{-y} e^{qz} \mathcal{F}_q^{-1}[\phi](z)dz \end{aligned}$$

Using the representation formula (3.4) for \mathcal{F}_q^{-1} , we obtain

$$\begin{aligned} \mathcal{L}_+[\mathcal{F}_q^{-1}[\phi]](y) \\ = \mathcal{F}^{-1}[\phi](y) - qe^{-qy} \int_{-\infty}^y e^{qz} \mathcal{F}^{-1}[1_+\phi](z)dz \\ - qe^{-qy} \int_{-y}^{\infty} e^{-qz} \mathcal{F}^{-1}[1_-\phi](z)dz \\ - qe^{qy} \int_y^0 e^{-qz} \mathcal{F}^{-1}[\phi](z)dz + q^2 e^{qy} \int_y^0 e^{-2qz} \left(\int_{-\infty}^z e^{qw} \mathcal{F}^{-1}[1_+\phi](w)dw \right) dz \\ + q^2 e^{qy} \int_y^0 e^{-2qz} \left(\int_{-z}^{\infty} e^{-qw} \mathcal{F}^{-1}[1_-\phi](w)dw \right) dz \\ - qe^{qy} \int_0^{-y} e^{qz} \mathcal{F}^{-1}[\phi](z)dz + q^2 e^{qy} \int_0^{-y} e^{2qz} \left(\int_{-\infty}^{-z} e^{qw} \mathcal{F}^{-1}[1_+\phi](w)dw \right) dz \\ + q^2 e^{qy} \int_0^{-y} e^{2qz} \left(\int_z^{\infty} e^{-qw} \mathcal{F}^{-1}[1_-\phi](w)dw \right) dz \\ \equiv G_1(x) + \cdots + G_9(x). \end{aligned}$$

Notice that for $y < 0$,

$$e^{qy} \int_y^0 e^{-2qz} \left(\int_{-\infty}^z e^{qw} f(w)dw \right) dz = e^{qy} \int_0^{-y} e^{2qz} \left(\int_{-\infty}^{-z} e^{qw} f(w)dw \right) dz$$

$$\begin{aligned}
&= \frac{1}{2q} e^{qy} \int_y^0 e^{-qz} f(z) dz + \frac{1}{2q} e^{-qy} \int_{-\infty}^y e^{qz} f(z) dz - \frac{1}{2q} e^{qy} \int_{-\infty}^0 e^{qz} f(z) dz, \\
&e^{qy} \int_y^0 e^{-2qz} \left(\int_{-z}^{\infty} e^{-qw} f(w) dw \right) dz = e^{qy} \int_0^{-y} e^{2qz} \left(\int_z^{\infty} e^{-qw} f(w) dw \right) dz \\
&= \frac{1}{2q} e^{qy} \int_0^{-y} e^{qz} f(z) dz + \frac{1}{2q} e^{-qy} \int_{-y}^{\infty} e^{-qz} f(z) dz - \frac{1}{2q} e^{qy} \int_0^{\infty} e^{-qz} f(z) dz.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&G_2(x) + G_5(x) + G_8(x) \\
&= qe^{qy} \int_y^0 e^{-qz} \mathcal{F}^{-1}[1_+\phi](z) dz - qe^{qy} \int_{-\infty}^0 e^{qz} \mathcal{F}^{-1}[1_+\phi](z) dz \\
&= -G_4(x) - qe^{qy} \int_y^0 e^{-qz} \mathcal{F}^{-1}[1_-\phi](z) dz - qe^{qy} \int_{-\infty}^0 e^{qz} \mathcal{F}^{-1}[1_+\phi](z) dz, \\
&G_3(x) + G_6(x) + G_9(x) \\
&= qe^{qy} \int_0^{-y} e^{qz} \mathcal{F}^{-1}[1_-\phi](z) dz - qe^{qy} \int_0^{\infty} e^{-qz} \mathcal{F}^{-1}[1_-\phi](z) dz \\
&= -G_7(x) - qe^{qy} \int_0^{-y} e^{qz} \mathcal{F}^{-1}[1_+\phi](z) dz - q \int_0^{\infty} e^{-qz} \mathcal{F}^{-1}[1_-\phi](z) dz.
\end{aligned}$$

Collecting the above identities, we obtain (3.6). By a similar argument as above we also obtain (3.7), which completes the proof of Lemma 3.1. \square

Acknowledgments. The author is partially supported by MEXT, Grant-in-Aid for Young Scientists (A) 25707004 and the Sumitomo Foundation, Basic Science Research Projects No. 120043.

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